Undetermined Coefficients another example

Ex2: Solve the i.v.p. using the **Undetermined Coefficients Method**

$$\vec{Y}\,' = \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}_{A} \vec{Y} + \underbrace{\begin{bmatrix} e^t \\ 2t \end{bmatrix}}_{G(t)}, \quad \vec{Y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Rewrite: $Y' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} Y + \underbrace{\begin{bmatrix} e^t \\ 0 \end{bmatrix}}_{G_1} + \underbrace{\begin{bmatrix} 0 \\ 2t \end{bmatrix}}_{G_2}$

Seperate

$$\begin{cases} Y' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} Y + \begin{bmatrix} e^t \\ 0 \end{bmatrix}_{G_1} \leftarrow Y_{p_1} = \begin{bmatrix} 0 \\ -\frac{1}{2}e^t \end{bmatrix} \\ Y' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} Y + \begin{bmatrix} 0 \\ 2t \end{bmatrix}_{G_2} \leftarrow Y_{p_2} = [\end{cases}$$

DNF?

① Find the Y_p for $Y' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} Y' + \underbrace{\begin{bmatrix} 0 \\ 2t \end{bmatrix}}_{G}$

② Guess $Y_p = t \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = t\vec{a} + \vec{b}$

③ Sub Y_p into $Y = AY + G_T \longrightarrow G = \begin{bmatrix} 0 \\ 2t \end{bmatrix} = \underbrace{t \begin{bmatrix} 0 \\ 2 \end{bmatrix}}_{t \cdot \vec{g}}$

$$(t\vec{a} + \vec{b})' = A(t\vec{a} + \vec{b}) + t \cdot \vec{g}$$

$$\vec{a} = t \cdot (A\vec{a}) + A\vec{b} + t \cdot \vec{g}$$
$$\vec{a} = t(A\vec{a} + \vec{g}) + A\vec{b}$$

Coeff of constant terms: $\vec{a} = A \cdot \vec{b}$ — ①

Coeff of t: $A\vec{a} + \vec{g} = 0$ — ②

② $A\vec{a} + \vec{g} = 0 \Rightarrow Aa = -\vec{g}$

$\Rightarrow \boxed{a = -A^{-1}\vec{g}}$

① $A \cdot \vec{b} = \vec{a} \Rightarrow \vec{b} = A^{-1} \cdot \vec{a}$

② $\vec{a} = -A^{-1}\vec{g} = -\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}^{-1} \cdot \begin{bmatrix}0\\ 2\end{bmatrix}$

$|A| = -3 \quad \Rightarrow \frac{1}{3}\begin{bmatrix}1 & -2\\ -2 & 1\end{bmatrix}\begin{bmatrix}0\\ 2\end{bmatrix} = \frac{1}{3}\begin{bmatrix}-4\\ 2\end{bmatrix} = \begin{bmatrix}-4/3\\ 2/3\end{bmatrix}$

$\vec{a}$

① $\vec{b} = A^{-1}\vec{a}$

$= -\frac{1}{3}\begin{bmatrix}1 & -2\\ -2 & 1\end{bmatrix}\begin{bmatrix}-4/3\\ 2/3\end{bmatrix} = \begin{bmatrix}8/9\\ -10/9\end{bmatrix}$

$Y_p = t\vec{a} + \vec{b} = t\begin{bmatrix}-4/3\\ 2/3\end{bmatrix} + \begin{bmatrix}8/9\\ -10/9\end{bmatrix}$

④ $Y = Y_{p_1} + Y_p$

$= \begin{bmatrix}0\\ -\frac{1}{2}e^{t}\end{bmatrix} + \underbrace{\left(t\begin{bmatrix}-4/3\\ 2/3\end{bmatrix} + \begin{bmatrix}8/9\\ -10/9\end{bmatrix}\right)}$

⑤ General Sol$_n$:

$Y = Y_c + Y_p \qquad \Phi_c$

$= \begin{bmatrix}e^{3t} & e^{-t}\\ e^{3t} & e^{-t}\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix} + Y_p$